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Type II and heterotic one loop string effective actions in four dimensions

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Abstract: We analyze the reduction to four dimensions of the $\mathcal{R}^4$ terms which are part of the ten-dimensional string effective actions, both at tree level and one loop. We show that there are two independent combinations of $\mathcal{R}^4$ present, at one loop, in the type IIA four dimensional effective action, which means they both have their origin in M-theory. The $d = 4$ heterotic effective action also has such terms. This contradicts the common belief that there is only one $\mathcal{R}^4$ term in four-dimensional supergravity theories, given by the square of the Bel-Robinson tensor. In pure $\mathcal{N} = 1$ supergravity this new $\mathcal{R}^4$ combination cannot be directly supersymmetrized, but we show that, when coupled to a scalar chiral multiplet (violating the U(1) R-symmetry), it emerges in the action after elimination of the auxiliary fields.

Keywords: Supersymmetric Effective Theories, Superspaces, Supergravity Models.

Contents

1. Introduction

String theories require higher order in α' corrections to their corresponding low energy supergravity effective actions. The leading type II string corrections are of order α'^3, and include $\mathcal{R}^4$ terms (the fourth power of the Riemann tensor), both at tree level and one loop [1, 2]. These $\mathcal{R}^4$ corrections are also present in the type I/heterotic effective actions [3] and in M-theory [4].

These string corrections to supergravity theories should obviously be supersymmetric. Unfortunately there is still no known way to compute these corrections in a manifestly supersymmetric way, although important progresses have been achieved. The supersymmetrization of these higher order string/M-theory terms has been a topic of research for a long time [5, 6].

After compactification to four dimensions, one obtains a supergravity theory, whose number $\mathcal{N}$ of supersymmetries and different matter couplings depend crucially on the manifold where the compactification is taken. Most of the times, in four dimensions the higher order terms are studied as part of the supergravity theories, either simple [7−9] or extended [10−13], and are therefore considered only from a supergravity point of view. These theories are believed to be divergent, and those are candidate counterterms. Their possible stringy origin, as higher order terms in string/M theory after compactification from ten/eleven dimensions, is often neglected. One of the reasons for that criterion is chronological: the study of the quantum properties of four dimensional supergravity theories started several years before superstring theories were found to be free of anomalies and taken as the main candidates to a unified theory of all the interactions. In higher dimensions the procedure has been different: the low-energy limits of superstring theories are the different ten-dimensional supergravity theories. People have studied higher order corrections to these theories most of the times in the context of string theory, which requires them to be supersymmetric.

Tacitly one makes the natural assumption that, when compactified, these higher order terms also emerge as corrections to the corresponding four-dimensional supergravity theories. But this does not necessarily need to be the case. The quantum behavior of these theories is still an active topic of research, and recent works claim that the maximal $\mathcal{N} = 8$ theory may even be ultraviolet finite [14, 15]. If that is the case, the $\mathcal{N} = 8$ higher order terms will not be necessary from a supergravity point of view, although they will still appear in the $\mathcal{N} = 8$ theory we obtain when we compactify type II superstrings on a six-dimensional torus. All the higher order terms considered are, from a supergravity point of view, *candidate* counterterms; it has never been explicitly shown that they indeed appear in the quantum effective actions with nonzero coefficients. Even in $\mathcal{N} < 8$ theories, it may eventually happen that some of these counterterms are not necessary as supergravity counterterms, but are needed as compactified string corrections.

From the known bosonic terms in the different α'-corrected string effective actions in ten dimensions, one should therefore determine precisely which terms should emerge in four dimensions for each compactification manifold, not worrying if they are needed in $d = 4$ supergravity. This is the goal of the present article, but here we restrict ourselves mainly to the order α'^3 $\mathcal{R}^4$ terms. We will also be mainly (but not strictly) concerned with the simplest toroidal compactifications; the reason is that the terms one gets are "universal", i.e. they must be present (possibly together with other moduli-dependent terms) no matter which compactification manifold we take.

The article is organized as follows. In section 2 we review the purely gravitational parts in the effective actions, up to order α'^3, of type IIA, IIB and heterotic strings, at tree level and one loop. In section 3 we analyze their dimensional reduction to $d = 4$. We show that there are two independent $\mathcal{R}^4$ terms in the four dimensional superstring effective action, although a classical result tells us that, of these terms, only the one which was previously known can be directly supersymmetrized. The supersymmetrization of the new $\mathcal{R}^4$ term gives rise to a new problem, which we address in $\mathcal{N} = 1$ supergravity in section 4 by considering the coupling of the new $\mathcal{R}^4$ term to a chiral multiplet in superspace.

2. String effective actions to order α'^3 in $d = 10$

The Riemann tensor admits, in d spacetime dimensions, the following decomposition in terms of the Weyl tensor $\mathcal{W}_{mnpq}$, the Ricci tensor $\mathcal{R}_{mn}$ and the Ricci scalar $\mathcal{R}$:

$$\begin{aligned}\mathcal{R}_{mnpq} = \mathcal{W}_{mnpq} &- \frac{1}{d-2}\left(g_{mp}\mathcal{R}_{nq} - g_{np}\mathcal{R}_{mq} + g_{nq}\mathcal{R}_{mp} - g_{mq}\mathcal{R}_{np}\right) \\ &+ \frac{1}{(d-1)(d-2)}\left(g_{mp}g_{nq} - g_{np}g_{mq}\right)\mathcal{R}.\end{aligned} \tag{2.1}$$

As proven in [16], in $d = 10$ dimensions, the critical dimension of superstring theories, there are seven independent real scalar polynomials made from four powers of the irreducible components of the Weyl tensor, which we label, according to [5], as $R_{41}, \ldots, R_{46}, A_7$. These polynomials are given by

$$\begin{aligned}
R_{41} &= \mathcal{W}_{mnpq}\mathcal{W}^{nrqt}\mathcal{W}_{rstu}\mathcal{W}^{smup}, \\
R_{42} &= \mathcal{W}_{mnpq}\mathcal{W}^{nrqt}\mathcal{W}^{ms}{}_{tu}\mathcal{W}_{sr}{}^{up}, \\
R_{43} &= \mathcal{W}_{mnpq}\mathcal{W}_{rs}{}^{pq}\mathcal{W}^{mn}{}_{tu}\mathcal{W}^{rstu}, \\
R_{44} &= \mathcal{W}_{mnpq}\mathcal{W}^{mnpq}\mathcal{W}_{rstu}\mathcal{W}^{rstu}, \\
R_{45} &= \mathcal{W}_{mnpq}\mathcal{W}^{nrpq}\mathcal{W}_{rstu}\mathcal{W}^{smtu}, \\
R_{46} &= \mathcal{W}_{mnpq}\mathcal{W}_{rs}{}^{pq}\mathcal{W}^{mr}{}_{tu}\mathcal{W}^{nstu}, \\
A_7 &= \mathcal{W}_{mn}{}^{pq}\mathcal{W}^{mt}{}_{pu}\mathcal{W}_{tr}{}^{ns}\mathcal{W}^{ur}{}_{qs}.
\end{aligned} \tag{2.2}$$

The superstring α'^3 effective actions are given in terms of two independent bosonic terms, from which two separate superinvariants are built [5, 17]. These terms are given, at linear order in the NS-NS gauge field B_{mn}, by:

$$\begin{aligned}
I_X &= t_8 t_8 \mathcal{R}^4 + \frac{1}{2}\varepsilon_{10} t_8 B \mathcal{R}^4, \\
I_Z &= -\varepsilon_{10}\varepsilon_{10}\mathcal{R}^4 + 4\varepsilon_{10} t_8 B \mathcal{R}^4.
\end{aligned} \tag{2.3}$$

Each t_8 tensor has eight free spacetime indices. It acts in four two-index antisymmetric tensors, as defined in [1, 2], where one can also find the precise index contractions. In terms of the seven fundamental polynomials $R_{41}, \ldots, R_{46}, A_7$ from (2.2), the purely gravitational parts of I_X and I_Z, which we denote by X and Z respectively, are given by [5]:

$$\begin{aligned}
X := t_8 t_8 \mathcal{W}^4 &= 192 R_{41} + 384 R_{42} + 24 R_{43} + 12 R_{44} - 192 R_{45} - 96 R_{46}, \\
\frac{1}{8} Z := -\frac{1}{8}\varepsilon_{10}\varepsilon_{10}\mathcal{W}^4 &= X + 192 R_{46} - 768 A_7.
\end{aligned} \tag{2.4}$$

For the heterotic string two extra terms Y_1 and Y_2 appear at order α'^3 at one loop level [5, 6, 17], the pure gravitational parts of which being given respectively by

$$\begin{aligned}
Y_1 := t_8 \left(\mathrm{tr}\mathcal{W}^2\right)^2 &= -4 R_{43} - 2 R_{44} + 16 R_{45} + 8 R_{46}, \\
Y_2 := t_8 \mathrm{tr}\mathcal{W}^4 &= 8 R_{41} + 16 R_{42} - 4 R_{45} - 2 R_{46}.
\end{aligned} \tag{2.5}$$

with $\mathrm{tr}\mathcal{W}^2 = \mathcal{W}_{mnpq}\mathcal{W}_{rs}{}^{qp}$, etc. Only three of these four invariants are independent because, as one may see, one has the relation $X = 24 Y_2 - 6 Y_1$.

To be precise, let's review the form of the purely gravitational superstring and heterotic effective actions in the string frame up to order α'^3. The perturbative terms occur at string tree and one loop levels; there are no higher loop contributions [4, 17 − 19].

The effective action of type IIB theory must be written, because of its well known $\mathrm{SL}(2,\mathbb{Z})$ invariance, as a product of a single linear combination of order α'^3 invariants and an overall function of the complexified coupling constant $\Omega = C^0 + i e^{-\phi}$, C^0 being the axion.

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This function accounts for perturbative (loop) and non-perturbative (D-instanton [18, 20]) string contributions. The perturbative part is given in the string frame by

$$\frac{1}{\sqrt{-g}}\mathcal{L}_{\rm IIB}\bigg|_{\alpha'^3} = -e^{-2\phi}\alpha'^3\frac{\zeta(3)}{3\times 2^{10}}\left(I_X-\frac{1}{8}I_Z\right)-\alpha'^3\frac{1}{3\times 2^{16}\pi^5}\left(I_X-\frac{1}{8}I_Z\right). \tag{2.6}$$

Type IIA theory has exactly the same term of order α'^3 as type IIB at tree level, but at one loop the sign in the coefficient of I_Z is changed when compared to type IIB:

$$\frac{1}{\sqrt{-g}}\mathcal{L}_{\rm IIA}\bigg|_{\alpha'^3} = -e^{-2\phi}\alpha'^3\frac{\zeta(3)}{3\times 2^{10}}\left(I_X-\frac{1}{8}I_Z\right)-\alpha'^3\frac{1}{3\times 2^{16}\pi^5}\left(I_X+\frac{1}{8}I_Z\right). \tag{2.7}$$

The reason for this sign flip is that at one string loop the relative GSO projection between the left and right movers is different for type IIA and type IIB, since these two theories have different chirality properties [21, 22].

Type II superstring theories only admit α'^3 and higher corrections because the corresponding sigma model is two and three-loop finite, as shown in [2]: ten dimensional $\mathcal{N}=2$ supersymmetry prevents these corrections. Heterotic string theories have $\mathcal{N}=1$ supersymmetry in ten dimensions, which allows corrections to the sigma model already at order α', including $\mathcal{R}^2$ corrections. These corrections come both from three-graviton scattering amplitudes and anomaly cancellation terms (the Green-Schwarz mechanism). The effective action is then given in the string frame, up to order α'^3 and neglecting the contributions of gauge fields, by

$$\begin{aligned}\frac{1}{\sqrt{-g}}\mathcal{L}_{\rm heterotic}\bigg|_{\alpha'+\alpha'^3} = & \; e^{-2\phi}\left[\frac{1}{16}\alpha'{\rm tr}\mathcal{R}^2+\frac{1}{2^9}\alpha'^3Y_1-\frac{\zeta(3)}{3\times 2^{10}}\alpha'^3\left(I_X-\frac{1}{8}I_Z\right)\right] \\ & -\alpha'^3\frac{1}{3\times 2^{14}\pi^5}\left(Y_1+4Y_2\right).\end{aligned} \tag{2.8}$$

For the type IIB theory only the combination $I_X-\frac{1}{8}I_Z$ is present in the effective action. For the type IIA and heterotic theories different combinations show up. The supersymmetrization of these terms has been the object of study in many articles [5, 6], although a complete understanding of the full supersymmetric effective actions is still lacking. Here we are more concerned with the number of independent superinvariants they would belong to. Because in every theory the $I_X-\frac{1}{8}I_Z$ term includes a transcendental factor $\zeta(3)$ (which is not shared by any other bosonic term at the same order in α'), it cannot be related to other bosonic terms by supersymmetry and requires its own superinvariant. This way in type IIA and heterotic string theories one then needs at least one $\mathcal{R}^4$ superinvariant for the tree level terms and another one for one loop.

Type IIA theory comes from compactification of M-theory on $\mathbb{S}^1$, but its tree level α'^3 terms vanish on the eleven-dimensional limit, as shown in [4]. Therefore the one-loop type IIA $\mathcal{R}^4$ term is the true compactification of the $d=11$ $\mathcal{R}^4$ term. In M-theory, there is only one $\mathcal{R}^4$ superinvariant. The existence of this term was shown in [23], using spinorial cohomology, and its coefficient was fixed using anomaly cancellation arguments. The full calculation, using pure spinor BRST cohomology, was carried out in [24], where it was shown that this term is indeed unique and its coefficient can be directly determined without using the anomaly cancellation argument.

For a more detailed review of the present knowledge of $\mathcal{R}^4$ terms in M-theory and supergravity, including a discussion of their supersymmetrization and related topics, see [25].

3. String effective actions to order α'^3 in $d = 4$

In this section we analyze the reduction to four dimensions of the effective actions considered in the previous section.

3.1 $\mathcal{R}^4$ terms in $d = 4$ from $d = 10$

It is interesting to check how many independent superinvariants one still has in four dimensions. In this case, the Weyl tensor can still be decomposed in its self-dual and antiself-dual parts[1]:

$$\mathcal{W}_{\mu\nu\rho\sigma} = \mathcal{W}^+_{\mu\nu\rho\sigma} + \mathcal{W}^-_{\mu\nu\rho\sigma}, \mathcal{W}^\mp_{\mu\nu\rho\sigma} := \frac{1}{2}\left(\mathcal{W}_{\mu\nu\rho\sigma} \pm \frac{i}{2}\varepsilon_{\mu\nu}{}^{\lambda\tau}\mathcal{W}_{\lambda\tau\rho\sigma}\right), \tag{3.1}$$

which have the following properties:

$$\mathcal{W}^+_{\mu\nu\rho\sigma}\mathcal{W}^{-\ \ \rho\sigma}_{\tau\lambda} = 0, \mathcal{W}^\pm_{\mu\nu\rho\sigma}\mathcal{W}^{\pm\nu\rho\sigma}_{\tau} = \frac{1}{4}g_{\mu\tau}\mathcal{W}^2_\pm. \tag{3.2}$$

Besides the usual Bianchi identities, the Weyl tensor in four dimensions obeys Schouten identities like this one:

$$\mathcal{W}^{\mu\nu}{}_{\rho\tau}\mathcal{W}_{\mu\nu\sigma\lambda} = \frac{1}{4}\left(g_{\rho\sigma}g_{\tau\lambda} - g_{\rho\lambda}g_{\tau\sigma}\right)\mathcal{W}^2 + 2\left(\mathcal{W}_{\rho\mu\nu\sigma}\mathcal{W}_\lambda{}^{\mu\nu}{}_\tau - \mathcal{W}_{\tau\mu\nu\sigma}\mathcal{W}_\lambda{}^{\mu\nu}{}_\rho\right). \tag{3.3}$$

Because of the given properties, the Bel-Robinson tensor, which can be shown to be totally symmetric, is given in four dimensions by

$$\mathcal{W}^+_{\mu\rho\nu\sigma}\mathcal{W}^{-\rho\ \sigma}_{\tau\ \lambda}.$$

In the van der Warden notation, using spinorial indices, the decomposition (3.1) is written as [26]

$$\mathcal{W}_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}} = -2\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{C}\dot{D}}\mathcal{W}_{ABCD} - 2\varepsilon_{AB}\varepsilon_{CD}\mathcal{W}_{\dot{A}\dot{B}\dot{C}\dot{D}} \tag{3.4}$$

with the totally symmetric $\mathcal{W}_{ABCD}, \mathcal{W}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ being given by (in the notation of [9])

$$\mathcal{W}_{ABCD} := -\frac{1}{8}\mathcal{W}^+_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}},\ \mathcal{W}_{\dot{A}\dot{B}\dot{C}\dot{D}} := -\frac{1}{8}\mathcal{W}^-_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\underline{\dot{A}\dot{B}}}\sigma^{\rho\sigma}_{\underline{\dot{C}\dot{D}}}.$$

Using this notation, calculations involving the Weyl tensor become much more simplified. The Bel-Robinson tensor is simply given by $\mathcal{W}_{ABCD}\mathcal{W}_{\dot{A}\dot{B}\dot{C}\dot{D}}$.

In reference [16] it is also shown that, in four dimensions, there are only two independent real scalar polynomials made from four powers of the Weyl tensor. Like in [9], these polynomials can be written, using the previous notation, as

$$\mathcal{W}^2_+\mathcal{W}^2_- = \mathcal{W}^{ABCD}\mathcal{W}_{ABCD}\mathcal{W}^{\dot{A}\dot{B}\dot{C}\dot{D}}\mathcal{W}_{\dot{A}\dot{B}\dot{C}\dot{D}}, \tag{3.5}$$

$$\mathcal{W}^4_+ + \mathcal{W}^4_- = \left(\mathcal{W}^{ABCD}\mathcal{W}_{ABCD}\right)^2 + \left(\mathcal{W}^{\dot{A}\dot{B}\dot{C}\dot{D}}\mathcal{W}_{\dot{A}\dot{B}\dot{C}\dot{D}}\right)^2. \tag{3.6}$$

[1] In the previous section, we used latin letters - $m, n, \ldots$ - to represent ten dimensional spacetime indices. From now on we will be only working with four dimensional spacetime indices which, to avoid any confusion, we represent by greek letters $\mu, \nu, \ldots$

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In particular, the seven polynomials $R_{41}, \ldots, R_{46}, A_7$ from (2.2), when computed directly in four dimensions (i.e. replacing the ten dimensional indices $m, n, \ldots$ by the four dimensional indices $\mu, \nu, \ldots$) should be expressed in terms of them. That is what we present in the following. For that we wrote each polynomial in the van der Warden notation, using (3.4), and we used some properties of the four dimensional Weyl tensor, like (3.2) and (3.3). This way we have shown that, *in four dimensions*,

$$\begin{aligned} R_{41} &= \frac{1}{24}\mathcal{W}_+^4 + \frac{1}{24}\mathcal{W}_-^4 - \frac{5}{8}\mathcal{W}_+^2\mathcal{W}_-^2, & R_{42} &= \frac{1}{12}\mathcal{W}_+^4 + \frac{1}{12}\mathcal{W}_-^4 + \frac{11}{8}\mathcal{W}_+^2\mathcal{W}_-^2, \\ R_{43} &= \frac{1}{6}\mathcal{W}_+^4 + \frac{1}{6}\mathcal{W}_-^4 - 4\mathcal{W}_+^2\mathcal{W}_-^2, & R_{44} &= \mathcal{W}_+^4 + \mathcal{W}_-^4 + 2\mathcal{W}_+^2\mathcal{W}_-^2, \\ R_{45} &= \frac{1}{4}\mathcal{W}_+^4 + \frac{1}{4}\mathcal{W}_-^4 + \frac{1}{2}\mathcal{W}_+^2\mathcal{W}_-^2, & R_{46} &= -\frac{1}{6}\mathcal{W}_+^4 - \frac{1}{6}\mathcal{W}_-^4 - \frac{3}{2}\mathcal{W}_+^2\mathcal{W}_-^2, \\ A_7 &= -\frac{1}{24}\mathcal{W}_+^4 - \frac{1}{24}\mathcal{W}_-^4 - \frac{1}{4}\mathcal{W}_+^2\mathcal{W}_-^2. \end{aligned} \tag{3.7}$$

Using the definitions (2.4), we have then

$$\begin{aligned} X &= 24\left(\mathcal{W}_+^4 + \mathcal{W}_-^4\right) + 384\mathcal{W}_+^2\mathcal{W}_-^2, \\ \frac{1}{8}Z &= 24\left(\mathcal{W}_+^4 + \mathcal{W}_-^4\right) + 288\mathcal{W}_+^2\mathcal{W}_-^2, \end{aligned} \tag{3.8}$$

or

$$X - \frac{1}{8}Z = 96\mathcal{W}_+^2\mathcal{W}_-^2, \tag{3.9}$$

$$X + \frac{1}{8}Z = 48\left(\mathcal{W}_+^4 + \mathcal{W}_-^4\right) + 672\mathcal{W}_+^2\mathcal{W}_-^2. \tag{3.10}$$

$X - \frac{1}{8}Z$ is the only combination of X and Z which in $d = 4$ does not contain (3.6), i.e. which contains only the square of the Bel-Robinson tensor (3.5). We find it extremely interesting that exactly this very same combination (or, to be precise, $I_X - \frac{1}{8}I_Z$) is, from (2.3), the only one which does not depend on the ten dimensional field B^{mn} and, therefore, due to its gauge invariance, is the only one that can appear in string theory at arbitrary loop order. This combination is indeed present at string tree level in every superstring theory, multiplied by a transcendental factor $\zeta(3)$, as we have seen in the previous section.

From (2.5) one also derives in $d = 4$:

$$Y_1 = 8\mathcal{W}_+^2\mathcal{W}_-^2, \tag{3.11}$$

$$Y_1 + 4Y_2 = \frac{X}{6} + 2Y_1 = 80\mathcal{W}_+^2\mathcal{W}_-^2 + 4\left(\mathcal{W}_+^4 + \mathcal{W}_-^4\right). \tag{3.12}$$

As seen in the previous section, for the type IIB theory only the combination $I_X - \frac{1}{8}I_Z$ (or $\mathcal{W}_+^2\mathcal{W}_-^2$ in $d = 4$) is present in the effective action (2.6). For the type IIA and heterotic theories different combinations show up. In these two cases, $\mathcal{W}_+^4 + \mathcal{W}_-^4$ shows up at string one loop level in the effective actions (2.7) and (2.8) of these theories when they are compactified to four dimensions. At string tree level, though, for all these theories in $d = 4$ only $\mathcal{W}_+^2\mathcal{W}_-^2$ shows up. This fact is quite remarkable, particularly for the heterotic theory, if we consider that the two different contributions $I_X - \frac{1}{8}I_Z$ and Y_1 in (2.8) have completely different origins.

3.2 Moduli-independent terms in $d = 4$ effective actions

All the terms we have been considering, when taken in the Einstein frame (which is the right frame for a supergravity analysis to be performed), are multiplied by an adequate power of $\exp(\phi)$. To be precise, consider an arbitrary term $I_i(\mathcal{R}, \mathcal{M})$ in the string frame lagrangian in d dimensions. $I_i(\mathcal{R}, \mathcal{M})$ is a function, with conformal weight w_i, of any given order in α', of the Riemann tensor $\mathcal{R}$ and any other fields - gauge fields, scalars, and also fermions - which we generically designate by $\mathcal{M}$. To pass from the string to the Einstein frame, we redefine the metric through a conformal transformation involving the dilaton, given by

$$\begin{aligned} g_{mn} &\to \exp\left(\frac{4}{d-2}\phi\right) g_{mn}, \\ \mathcal{R}_{mn}{}^{pq} &\to \exp\left(-\frac{4}{d-2}\phi\right) \widetilde{\mathcal{R}}_{mn}{}^{pq}, \end{aligned} \tag{3.13}$$

with $\widetilde{\mathcal{R}}_{mn}{}^{pq} = \mathcal{R}_{mn}{}^{pq} - \delta_{[m}{}^{[p} \nabla_{n]} \nabla^{q]} \phi$. The transformation above takes $I_i(\mathcal{R}, \mathcal{M})$ to $e^{\frac{4}{d-2} w_i \phi} I_i(\widetilde{\mathcal{R}}, \mathcal{M})$. After considering all the dilaton couplings and the effect of the conformal transformation on the metric determinant factor $\sqrt{-g}$, the string frame lagrangian

$$\frac{1}{2}\sqrt{-g}\, \mathrm{e}^{-2\phi}\Big(-\mathcal{R} + 4\,(\partial^m \phi)\, \partial_m \phi + \sum_i I_i(\mathcal{R}, \mathcal{M})\Big) \tag{3.14}$$

is converted into the Einstein frame lagrangian

$$\frac{1}{2}\sqrt{-g}\left(-\mathcal{R} - \frac{4}{d-2}\,(\partial^m \phi)\, \partial_m \phi + \sum_i \mathrm{e}^{\frac{4}{d-2}(1+w_i)\phi} I_i(\widetilde{\mathcal{R}}, \mathcal{M})\right). \tag{3.15}$$

We finish this section by writing, for later reference, the effective actions (2.6), (2.7), (2.8) in four dimensions, in the Einstein frame (considering only terms which are simply powers of the Weyl tensor, without any other fields except their couplings to the dilaton, and introducing the $d = 4$ gravitational coupling constant κ):

$$\frac{\kappa^2}{\sqrt{-g}} \mathcal{L}_{\mathrm{IIB}}\bigg|_{\mathcal{R}^4} = -\frac{\zeta(3)}{32} e^{-6\phi} \alpha'^3 \mathcal{W}_+^2 \mathcal{W}_-^2 - \frac{1}{2^{11}\pi^5} e^{-4\phi} \alpha'^3 \mathcal{W}_+^2 \mathcal{W}_-^2, \tag{3.16}$$

$$\begin{aligned} \frac{\kappa^2}{\sqrt{-g}} \mathcal{L}_{\mathrm{IIA}}\bigg|_{\mathcal{R}^4} &= -\frac{\zeta(3)}{32} e^{-6\phi} \alpha'^3 \mathcal{W}_+^2 \mathcal{W}_-^2 \\ &\quad \frac{1}{2^{12}\pi^5} e^{-4\phi} \alpha'^3 \left[\left(\mathcal{W}_+^4 + \mathcal{W}_-^4\right) + 224 \mathcal{W}_+^2 \mathcal{W}_-^2\right], \end{aligned} \tag{3.17}$$

$$\begin{aligned} \frac{\kappa^2}{\sqrt{-g}} \mathcal{L}_{\mathrm{het}}\bigg|_{\mathcal{R}^2 + \mathcal{R}^4} &= -\frac{1}{16} e^{-2\phi} \alpha' \left(\mathcal{W}_+^2 + \mathcal{W}_-^2\right) + \frac{1}{64}\left(1 - 2\zeta(3)\right) e^{-6\phi} \alpha'^3 \mathcal{W}_+^2 \mathcal{W}_-^2 \\ &\quad - \frac{1}{3 \times 2^{12}\pi^5} e^{-4\phi} \alpha'^3 \left[\left(\mathcal{W}_+^4 + \mathcal{W}_-^4\right) + 20 \mathcal{W}_+^2 \mathcal{W}_-^2\right]. \end{aligned} \tag{3.18}$$

Here one must refer that these are only the moduli-independent terms of these effective actions. Strictly speaking these are not moduli-independent terms, since they are all multiplied by the volume of the compactification manifold (a factor we omitted for simplicity).

But they are always present, no matter which compactification is taken. The complete action, for every different compactification manifold, includes many moduli-dependent terms which we do not consider here.

A complete study of the heterotic string moduli dependent terms, but only for $\alpha' = 0$ and for a $\mathbb{T}^6$ compactification, can be seen in [27]. The tree level and one loop contributions to the four graviton amplitude, for a compactification on an n-dimensional torus $\mathbb{T}^n$ of ten dimensional type IIA/IIB string theories, can be found in [20].

A detailed study of these moduli-dependent $\mathcal{R}^4$ terms, at string tree level and one loop, for type IIA and IIB superstrings, for several compactification manifolds preserving different ammounts of supersymmetry, is available in [28]. In many cases one must consider extra contributions to the effective action coming from string winding modes and worldsheet instantons. For the particularly simple but illustrative case of an $\mathbb{S}^1$ compactification (presented in detail in [20, 28]), the tree level terms for both type IIA and IIB theories are trivial: they are simply multiplied by the volume $2\pi R$. At one loop level, one gets terms proportional to the compactification radius R; by applying T-duality to these terms, one gets other terms proportional to $\frac{\alpha'}{R}$. This way one gets the term $X + \frac{1}{8}Z$, in $d = 9$, even for type IIB effective action (in this case, only at a higher order in α'). The same is true in $d = 4$, for more complicated compactification manifolds.

To conclude, for any $d = 4$ compactification of heterotic or superstring theories one has, in the respective effective action, the two different $d = 4$ $\mathcal{R}^4$ terms (3.5) and (3.6), multiplied by a corresponding dilaton factor and maybe some moduli terms. This is the most important result for the rest of this paper. From now on we will be concerned with the supersymmetrization of these terms.

4. $\mathcal{R}^4$ terms and $d = 4$ supersymmetry

Up to now, we have only been considering bosonic terms for the effective actions, but we are interested in their full supersymmetric completion in $d = 4$. In general each superinvariant consists of a leading bosonic term and its supersymmetric completion, given by a series of terms with fermions. In this work we are particularly focusing on $\mathcal{R}^4$ terms.

4.1 Some known results

It has been known for a long time that the square of the Bel-Robinson tensor $\mathcal{W}_+^2\mathcal{W}_-^2$ can be made supersymmetric, in simple [7, 8] and extended [10, 12, 13] four dimensional supergravity. For the term $\mathcal{W}_+^4 + \mathcal{W}_-^4$ there is a "no-go theorem", based on $\mathcal{N} = 1$ chirality arguments [29]: for a polynomial $I(\mathcal{W})$ of the Weyl tensor to be supersymmetrizable, each one of its terms must contain equal powers of $\mathcal{W}^+_{\mu\nu\rho\sigma}$ and $\mathcal{W}^-_{\mu\nu\rho\sigma}$. The whole polynomial must then vanish when either $\mathcal{W}^+_{\mu\nu\rho\sigma}$ or $\mathcal{W}^-_{\mu\nu\rho\sigma}$ do. The only exception is $\mathcal{W}^2 = \mathcal{W}_+^2 + \mathcal{W}_-^2$, which in $d = 4$ is part of the Gauss-Bonnet topological term and is automatically supersymmetric.

But the new term (3.6) is part of the heterotic and type IIA effective actions at one loop which must be supersymmetric, even after compactification to $d = 4$. One must then find out how this term can be made supersymmetric, circumventing the $\mathcal{N}=1$ chirality argument from [29]. That is our main goal in this paper.

One must keep in mind the assumptions in which it was derived, namely the preservation by the supersymmetry transformations of R-symmetry which, for $\mathcal{N}=1$, corresponds to U(1) and is equivalent to chirality. That is true for pure $\mathcal{N}=1$ supergravity, but to this theory and to most of the extended supergravity theories (except $\mathcal{N}=8$) one may add matter couplings and extra terms which violate U(1) R-symmetry and yet can be made supersymmetric, inducing corrections to the supersymmetry transformation laws which do not preserve U(1) R-symmetry.

Since the article [29] only deals with the term (3.6) by itself, one can consider extra couplings to it and only then try to supersymmetrize. These couplings could eventually (but not necessarily) break U(1) R-symmetry. This procedure is very natural, taking into account the scalar couplings that multiply (3.6) in the actions (3.17), (3.18).

Considering couplings to other multiplets and breaking U(1) may be possible in $\mathcal{N}=4$ supergravity, for $\mathbb{T}^6$ compactifications of heterotic strings, but $\mathcal{N}=1$ supergravity has the advantage of being much less restrictive than its extended counterparts. To our purposes, the simplest and most obvious choice of coupling is to $\mathcal{N}=1$ chiral multiplets. That is what we do in the following subsection.

4.2 $\mathcal{W}_+^4+\mathcal{W}_-^4$ in $\mathcal{N}=1$ matter-coupled supergravity

The $\mathcal{N}=1$ supergravity multiplet is very simple. What also makes this theory easier is the existence of several different full off-shell formulations. We work in standard "old minimal" supergravity, having as auxiliary fields a vector $A_{A\dot{A}}$, a scalar M and a pseudoscalar N, given as $\theta=0$ components of superfields $G_{A\dot{A}}, R, \overline{R}$:[2]

$$G_{A\dot{A}}\big| = \frac{1}{3}A_{A\dot{A}},\, \overline{R}\big| = 4\,(M+iN)\,,\, R\big| = 4\,(M-iN)\,. \tag{4.1}$$

Besides there is a chiral superfield W_{ABC} and its hermitian conjugate $W_{\dot{A}\dot{B}\dot{C}}$, which together at $\theta=0$ constitute the field strength of the gravitino. The Weyl tensor shows up as the first θ term: in the notation of (3.4), at the linearized level,

$$\nabla_{\underline{D}}W_{\underline{ABC}}\big| = \mathcal{W}_{ABCD}+\ldots \tag{4.2}$$

$\mathcal{W}_+^4+\mathcal{W}_-^4$ is proportional to the $\theta=0$ term of $\left(\nabla^2W^2\right)^2+$h.c., which cannot result from a superspace integration. This whole term itself is U(1) R-symmetric, like $\nabla_{\underline{D}}W_{\underline{ABC}}$; indeed, the components of the Weyl tensor are U(1) R-neutral, according to the weights [9]

$$\nabla_A\mapsto +1, R\mapsto +2, G_m\mapsto 0, W_{ABC}\mapsto -1.$$

This way, as expected, one needs some extra coupling to (3.6) in order to break U(1) R-symmetry. We can use the fact that there are many more matter fields with its origin in string theory and many different matter multiplets to which one can couple the $\mathcal{N}=1$ supergravity multiplet in order to build superinvariants. This way we hope to find some

[2]The $\mathcal{N}=1$ superspace conventions are exactly the same as in [8, 9].

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coupling which breaks U(1) R-symmetry and simultaneously supersymmetrizes (3.6), which could result from the elimination of the matter auxiliary fields.

Having this in mind, we consider a chiral multiplet, represented by a chiral superfield $\mathbf{\Phi}$ (we could take several chiral multiplets Φ_i, but we restrict ourselves to one for simplicity), and containing a scalar field $\Phi = \mathbf{\Phi}|$, a spin$-\frac{1}{2}$ field $\nabla_A \mathbf{\Phi}|$, and an auxiliary field $F = -\frac{1}{2}\nabla^2\mathbf{\Phi}\big|$. This superfield and its hermitian conjugate couple to $\mathcal{N}=1$ supergravity in its simplest version through a superpotential

$$P(\mathbf{\Phi}) = d + a\mathbf{\Phi} + \frac{1}{2}m\mathbf{\Phi}^2 + \frac{1}{3}g\mathbf{\Phi}^3 \tag{4.3}$$

and a Kähler potential $K\left(\mathbf{\Phi},\overline{\mathbf{\Phi}}\right) = -\frac{3}{\kappa^2}\ln\left(-\frac{\Omega\left(\mathbf{\Phi},\overline{\mathbf{\Phi}}\right)}{3}\right)$, with $\Omega\left(\mathbf{\Phi},\overline{\mathbf{\Phi}}\right)$ given by

$$\Omega\left(\mathbf{\Phi},\overline{\mathbf{\Phi}}\right) = -3 + \mathbf{\Phi}\overline{\mathbf{\Phi}} + c\mathbf{\Phi} + \overline{c}\overline{\mathbf{\Phi}}. \tag{4.4}$$

In order to include the term (3.6), we take the following effective action:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{6\kappa^2}\int E\left[\Omega\left(\mathbf{\Phi},\overline{\mathbf{\Phi}}\right) + \alpha'^3\left(b\mathbf{\Phi}\left(\nabla^2W^2\right)^2 + \overline{b}\overline{\mathbf{\Phi}}\left(\overline{\nabla}^2\overline{W}^2\right)^2\right)\right]d^4\theta \\ &\quad -\frac{2}{\kappa^2}\left(\int \epsilon P(\mathbf{\Phi})\, d^2\theta + \text{h.c.}\right) \\ &= \frac{1}{4\kappa^2}\int \epsilon\Bigg[\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\left(\Omega\left(\mathbf{\Phi},\overline{\mathbf{\Phi}}\right) + \alpha'^3\left(b\mathbf{\Phi}\left(\nabla^2W^2\right)^2 + \overline{b}\overline{\mathbf{\Phi}}\left(\overline{\nabla}^2\overline{W}^2\right)^2\right)\right) \\ &\qquad\qquad -8P(\mathbf{\Phi})\Bigg]d^2\theta + \text{h.c.}\end{aligned} \tag{4.5}$$

E is the superdeterminant of the supervielbein; ϵ is the chiral density. The $\Omega\left(\mathbf{\Phi},\overline{\mathbf{\Phi}}\right)$ and $P(\mathbf{\Phi})$ terms represent the most general renormalizable coupling of a chiral multiplet to pure supergravity [30]; the extra terms represent higher-order corrections. Of course (4.5) is meant as an effective action and therefore does not need to be renormalizable.

The component expansion of this action may be found using the explicit θ expansions for ϵ and ∇^2W^2 given in [9]. From (4.2), we have

$$\nabla^2W^2\big| = -2\mathcal{W}_+^2 + \dots \tag{4.6}$$

It is well known that an action of this type in pure supergravity (without the higher-order corrections) will give rise, in x-space, to a leading term given by $\frac{1}{6\kappa^2}e\,\Omega|\,\mathcal{R}$ instead of the usual $-\frac{1}{2\kappa^2}e\mathcal{R}$.[3] In order to remove the extra $\Phi\mathcal{R}$ terms in $\frac{1}{6\kappa^2}e\,\Omega|\,\mathcal{R}$, one takes a $\Phi,\overline{\Phi}$-dependent conformal transformation [30]; if one also wants to remove the higher order $\Phi\mathcal{R}$ terms, this conformal transformation must be α'-dependent. Here we are only interested in obtaining the supersymmetrization of $\mathcal{W}_+^4 + \mathcal{W}_-^4$; therefore we will not be concerned with the Ricci terms of any order.

[3] As usual in supergravity theories we work with the vielbein and not with the metric. Therefore, here we write e, the determinant of the vielbein, instead of $\sqrt{-g}$.

If one expands (4.5) in components, one does not directly get (3.6), but one should look at the auxiliary field sector. Because of the presence of the higher-derivative terms, the auxiliary field from the original conformal supermultiplet A_m also gets higher derivatives in its equation of motion, and therefore it cannot be simply eliminated [8, 12]. Here we only consider the much simpler terms which include the chiral multiplet auxiliary field F. Take the superfields

$$\tilde{\mathbf{C}} = c + \alpha'^3 b \left(\nabla^2 W^2\right)^2, \widetilde{\Omega}\left(\mathbf{\Phi}, \overline{\mathbf{\Phi}}, \tilde{\mathbf{C}}, \overline{\tilde{\mathbf{C}}}\right) = -3 + \mathbf{\Phi}\overline{\mathbf{\Phi}} + \tilde{\mathbf{C}}\mathbf{\Phi} + \overline{\tilde{\mathbf{C}}}\overline{\mathbf{\Phi}}, \tag{4.7}$$

so that the action (4.5) becomes

$$\frac{1}{4\kappa^2}\int \epsilon \left[\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\widetilde{\Omega}\left(\mathbf{\Phi}, \overline{\mathbf{\Phi}}, \tilde{\mathbf{C}}, \overline{\tilde{\mathbf{C}}}\right) - 8P\left(\mathbf{\Phi}\right)\right] d^2\theta + \text{h.c.} \tag{4.8}$$

and all the α'^3 corrections considered in it become implicitly included in $\widetilde{\Omega}\left(\mathbf{\Phi}, \overline{\mathbf{\Phi}}, \tilde{\mathbf{C}}, \overline{\tilde{\mathbf{C}}}\right)$ through $\tilde{\mathbf{C}}, \overline{\tilde{\mathbf{C}}}$. We also define $\tilde{C} = \tilde{\mathbf{C}}|$ and the functional derivative $P_{\mathbf{\Phi}} = \partial P/\partial \mathbf{\Phi}$. From now on, we will work in x-space and assume there is no confusion between the superfield functionals $\widetilde{\Omega}\left(\mathbf{\Phi}, \overline{\mathbf{\Phi}}, \tilde{\mathbf{C}}, \overline{\tilde{\mathbf{C}}}\right)$, $P\left(\mathbf{\Phi}\right)$, $P_{\mathbf{\Phi}}$ and their corresponding x-space functionals $\widetilde{\Omega}\left(\Phi, \overline{\Phi}, \tilde{C}, \overline{\tilde{C}}\right)$, $P\left(\Phi\right)$, P_{Φ}. The terms we are looking for are given by [30]

$$\begin{aligned}\kappa^2\mathcal{L}_{F,\overline{F}} = {} & \frac{1}{9}e\widetilde{\Omega}\left(\Phi, \overline{\Phi}, \tilde{C}, \overline{\tilde{C}}\right)\left|M - iN - \frac{3}{\widetilde{\Omega}\left(\Phi, \overline{\Phi}, \tilde{C}, \overline{\tilde{C}}\right)}\left(\Phi + \overline{\tilde{C}}\right)F\right|^2 \\ & - e\frac{3 + \tilde{C}\overline{\tilde{C}}}{\widetilde{\Omega}^2\left(\Phi, \overline{\Phi}, \tilde{C}, \overline{\tilde{C}}\right)}F\overline{F} + e\tilde{P}_{\Phi}F + e\overline{\tilde{P}}_{\overline{\Phi}}\overline{F}.\end{aligned} \tag{4.9}$$

This equation would be exact, with $\tilde{P}_{\Phi} = P_{\Phi}$ and $\overline{\tilde{P}}_{\overline{\Phi}} = \overline{P}_{\overline{\Phi}}$, if we were only considering the $\theta = 0$ components of $\tilde{\mathbf{C}}, \overline{\tilde{\mathbf{C}}}$. But, of course (as it is clear from (4.5)), coupled to F we will have $\nabla_{\dot{A}}\left(\nabla^2 W^2\right)^2$ and $\overline{\nabla}^2\left(\nabla^2 W^2\right)^2$ terms (and $\nabla_A\left(\overline{\nabla}^2\overline{W}^2\right)^2$ and $\nabla^2\left(\overline{\nabla}^2\overline{W}^2\right)^2$ terms coupled to $\bar{F}$). These terms will not play any role for our purpose (which is to show that there exists a supersymmetric lagrangian which contains (3.6), and not necessarily to compute it in full), and therefore we do not compute them explicitly. We write them in (4.9) because we include them in $\tilde{P}_{\Phi}$, through the definition (analogous for $\overline{\tilde{P}}_{\overline{\Phi}}$)

$$\tilde{P}_{\Phi} = P_{\Phi} + \left(\nabla_{\dot{A}}\tilde{\mathbf{C}} + \overline{\nabla}^2\tilde{\mathbf{C}}\,\text{terms}\right).$$

The first term in (4.9) contains the well known term $-\frac{1}{3}e\left(M^2 + N^2\right)$ from "old minimal" supergravity. Because the auxiliary fields M, N belong to the chiral compensating multiplet, their field equation should be algebraic, despite the higher derivative corrections [8, 12]. That calculation should still require some effort; plus, those M, N auxiliary fields should not generate *by themselves* terms which violate U(1) R-symmetry: these terms should only occur through the elimination of $F, \bar{F}$. This is why we will only be concerned with these auxiliary fields, which therefore can be easily eliminated through their field equation

$$\left(\frac{\left(\overline{\Phi}+\tilde{C}\right)\left(\Phi+\overline{\tilde{C}}\right)}{\widetilde{\Omega}\left(\Phi,\overline{\Phi},\tilde{C},\overline{\tilde{C}}\right)}-\frac{3+\tilde{C}\overline{\tilde{C}}}{\widetilde{\Omega}^{2}\left(\Phi,\overline{\Phi},\tilde{C},\overline{\tilde{C}}\right)}\right)F=-\overline{\tilde{P}}_{\overline{\Phi}}-\frac{1}{3}\left(\overline{\Phi}+\tilde{C}\right)(M-iN)\,.$$

Replacing $F,\bar{F}$ in $\mathcal{L}_{F,\overline{F}}$, one gets

$$\kappa^{2}\mathcal{L}_{F,\overline{F}}=-e\frac{\tilde{P}_{\Phi}\overline{\tilde{P}}_{\overline{\Phi}}\widetilde{\Omega}^{2}\left(\Phi,\overline{\Phi},\tilde{C},\overline{\tilde{C}}\right)}{\left(\overline{\Phi}+\tilde{C}\right)\left(\Phi+\overline{\tilde{C}}\right)\widetilde{\Omega}\left(\Phi,\overline{\Phi},\tilde{C},\overline{\tilde{C}}\right)-\left(\tilde{C}\overline{\tilde{C}}+3\right)}+M,N\text{ terms.}\qquad(4.10)$$

This is a nonlocal, nonpolynomial action. Since we take it as an effective action, we can expand it in powers of the fields $\Phi,\overline{\Phi}$, but also in powers of $\tilde{C},\overline{\tilde{C}}$. These last fields contain both the couplings of $\mathbf{\Phi}$ to supergravity c and the string parameter α'; expanding in these fields is equivalent to expanding in a certain combination of these parameters. Here one should notice that we are only considering up to α'^3 terms. If we wanted to consider higher (than α'^3) order corrections, together with these we should also have included *a priori* in (4.5) the leading higher order corrections, which should be independently supersymmetrized. Considering solely the higher than α'^3 order corrections coming directly from the elimination of (any of) the auxiliary fields from the α'^3 effective action (4.5) would be misleading. The correct expansion of (4.5) to take, in the first place, is in α'^3. That is what we do in the following, after replacing $\tilde{C},\overline{\tilde{C}}$ by their explicit superfield expressions given by (4.7) and taking $\theta=0$. We also exclude the M,N contributions and the higher θ terms from $\tilde{\mathbf{C}},\overline{\tilde{\mathbf{C}}}$ in $\tilde{P}_{\Phi},\overline{\tilde{P}}_{\overline{\Phi}}$, for the reasons mentioned before: they are not significant for the term we are looking for. The resulting lagrangian we get (which we still call $\mathcal{L}_{F,\overline{F}}$ to keep its origin clear, although it is not anymore the complete lagrangian resulting from the elimination of $F,\bar{F}$) is

$$\begin{aligned}\kappa^{2}\mathcal{L}_{F,\overline{F}}=&-e\frac{P_{\Phi}\overline{P}_{\overline{\Phi}}\Omega^{2}\left(\Phi,\overline{\Phi}\right)}{\left(\overline{\Phi}+c\right)\left(\Phi+\overline{c}\right)\Omega\left(\Phi,\overline{\Phi}\right)-\left(c\overline{c}+3\right)}\qquad(4.11)\\&+\alpha'^{3}\frac{eP_{\Phi}\overline{P}_{\overline{\Phi}}\Omega\left(\Phi,\overline{\Phi}\right)}{\left(\left(\overline{\Phi}+c\right)\left(\Phi+\overline{c}\right)\Omega\left(\Phi,\overline{\Phi}\right)-\left(c\overline{c}+3\right)\right)^{2}}\left[-2\left(b\Phi\left(\nabla^{2}W^{2}\right)^{2}\Big|\right.\right.\\&+\left.\overline{b}\overline{\Phi}\left(\overline{\nabla}^{2}\overline{W}^{2}\right)^{2}\Big|\right)\left(\left(\overline{\Phi}+c\right)\left(\Phi+\overline{c}\right)\Omega\left(\Phi,\overline{\Phi}\right)-\left(c\overline{c}+3\right)\right)\\&+\Omega\left(\Phi,\overline{\Phi}\right)\left(-b\overline{c}\Phi\left(\nabla^{2}W^{2}\right)^{2}\Big|-\overline{b}c\overline{\Phi}\left(\overline{\nabla}^{2}\overline{W}^{2}\right)^{2}\Big|\right.\\&+\left(\overline{\Phi}+c\right)\left(\Phi+\overline{c}\right)\left(b\Phi\left(\nabla^{2}W^{2}\right)^{2}\Big|+\overline{b}\overline{\Phi}\left(\overline{\nabla}^{2}\overline{W}^{2}\right)^{2}\Big|\right)\\&+\left.\left.\Omega\left(\Phi,\overline{\Phi}\right)\left(b\left(\overline{c}+\Phi\right)\left(\nabla^{2}W^{2}\right)^{2}\Big|+\overline{b}\left(c+\overline{\Phi}\right)\left(\overline{\nabla}^{2}\overline{W}^{2}\right)^{2}\Big|\right)\right)\right]+\ldots\end{aligned}$$

If we look at the last line of the previous equation, we can already identify the term we are looking for. This is still a nonlocal, nonpolynomial action, which we expand now in powers of the fields $\Phi, \overline{\Phi}$ coming from the denominators and the $P_\Phi \overline{P}_{\overline{\Phi}}$ factors. We obtain

$$
\begin{aligned}
\kappa^2 \mathcal{L}_{F,\overline{F}} = & -15e\frac{(3+c\overline{c})}{(3+4c\overline{c})^2}\left(m\overline{a}\Phi + \overline{m}a\overline{\Phi}\right)\left(c\Phi + \overline{c}\overline{\Phi}\right) \\
& + e\frac{2c^3\overline{c}^3 + 60c^2\overline{c}^2 + 117c\overline{c} - 135}{(3+4c\overline{c})^3} a\overline{a}\Phi\overline{\Phi} - 36\alpha'^3 e\left(b\overline{c}\left(\nabla^2 W^2\right)^2\right| \\
& \left. + \overline{b}c\left(\overline{\nabla}^2\overline{W}^2\right)^2\right|\Bigg) \frac{a\overline{a} + m\overline{a}\Phi + \overline{m}a\overline{\Phi} + g\overline{a}\Phi^2 + \overline{g}a\overline{\Phi}^2 + m\overline{m}\Phi\overline{\Phi}}{(3+4c\overline{c})^2} \\
& - 3\alpha'^3 a\overline{a}\frac{74c^2\overline{c}^2 + 192c\overline{c} - 657}{(3+4c\overline{c})^4}\Phi\overline{\Phi}\left(b\overline{c}\left(\nabla^2 W^2\right)^2\right| + \overline{b}c\left.\left(\overline{\nabla}^2\overline{W}^2\right)^2\right|\Bigg) \\
& + 15\alpha'^3 e\frac{a\overline{a} + m\overline{a}\Phi + \overline{m}a\overline{\Phi}}{(3+4c\overline{c})^3}\left[\left(\overline{c}^2\left(21+4c\overline{c}\right)\overline{\Phi} + \left(-9+6c\overline{c}\right)\Phi\right) b\left(\nabla^2 W^2\right)^2\right| \\
& + \left(c^2\left(21+4c\overline{c}\right)\Phi + \left(-9+6c\overline{c}\right)\overline{\Phi}\right)\overline{b}\left.\left(\overline{\nabla}^2\overline{W}\right)^2\right|\Bigg] + \ldots
\end{aligned}
\tag{4.12}
$$

This way we are able to supersymmetrize $\mathcal{W}_+^4 + \mathcal{W}_-^4$, although we had to introduce a coupling to a chiral multiplet. These multiplets show up after $d = 4$ compactifications of superstring and heterotic theories and truncation to $\mathcal{N} = 1$ supergravity [31]. Since from (4.6) the factor in front of $\mathcal{W}_+^4$ (resp. $\mathcal{W}_-^4$) in (4.12) is given by $\frac{72b\overline{c}a\overline{a}}{(3+4c\overline{c})^2}$ (resp. $\frac{72\overline{b}ca\overline{a}}{(3+4c\overline{c})^2}$), for this supersymmetrization to be effective, the factors a from $P(\Phi)$ in (4.3) and c from $\Omega\left(\Phi, \overline{\Phi}\right)$ in (4.4) (and of course b from (4.5)) must be nonzero.

The action (4.12) includes the $\mathcal{N} = 1$ supersymmetrization of $\mathcal{W}_+^4 + \mathcal{W}_-^4$, but without any coupling to a scalar field or only with couplings to powers of the scalar field from the chiral multiplet, which may be seen as compactification moduli. But, as one can see from (3.17), (3.18), this term should be coupled to powers of the dilaton. It is well known [31] that in $\mathcal{N} = 1$ supergravity the dilaton is part of a linear multiplet, together with an antisymmetric tensor field and a Majorana fermion. One must then work out the coupling to supergravity of the linear and chiral multiplets. As usual one starts from conformal supergravity and obtain Poincaré supergravity by coupling to compensator multiplets which break superconformal invariance through a gauge fixing condition. When there are only chiral multiplets coupled to supergravity [30], this gauge fixing condition can be generically solved, so that a lagrangian has been found for an arbitrary coupling of the chiral multiplets. In the presence of a linear multiplet, there is no such a generic solution of the gauge fixing condition, which must be solved case by case. Therefore, there is no generic lagrangian for the coupling of supergravity to linear multiplets. We shall not consider this problem here, like we did not in [8, 9]. In both cases we were only interested in studying the $\mathcal{N} = 1$ supersymmetrization of the two different $d = 4$ $\mathcal{R}^4$ terms. The coupling of a linear multiplet to these terms can be determined following the procedure in [32].

4.3 $\mathcal{W}_+^4 + \mathcal{W}_-^4$ in extended supergravity

$\mathcal{W}_+^4 + \mathcal{W}_-^4$ must also arise in extended $d = 4$ supergravity theories, for the reasons we saw, but the "no-go" result of ([29]) should remain valid, since it was obtained for

$\mathcal{N} = 1$ supergravity, which can always be obtained by truncating any extended theory. For extended supergravities, the chirality argument should be replaced by preservation by supergravity transformations of U(1), which is a part of R-symmetry.

$\mathcal{N} = 2$ supersymmetrization of $\mathcal{W}_+^4 + \mathcal{W}_-^4$ should work in a way similar to what we saw for $\mathcal{N} = 1$. $\mathcal{N} = 2$ chiral superfields must be Lorentz and SU(2) scalars but they can have an arbitrary U(1) weight, which allows supersymmetric U(1) breaking couplings.

A similar result should be more difficult to implement for $\mathcal{N} \geq 3$, because there are no generic chiral superfields. Still, there are other multiplets than the Weyl, which one can consider in order to couple to $\mathcal{W}_+^4 + \mathcal{W}_-^4$ and allow for its supersymmetrization. The only exception is $\mathcal{N} = 8$ supergravity, which only allows for the Weyl multiplet. $\mathcal{N} = 8$ supersymmetrization of $\mathcal{W}_+^4 + \mathcal{W}_-^4$ should therefore be a very difficult problem, which we expect to study in a future work.

Related to this is the issue of possible finiteness of $\mathcal{N} = 8$ supergravity, which has been a recent topic of research. A linearized three-loop candidate (the square of the Bel-Robinson tensor) has been presented in [10]. But recent works [14] show that there is no three-loop divergence (which includes the two $\mathcal{R}^4$ terms). Power-counting analysis from unitarity cutting-rule techniques predicted the lowest counterterm to appear at least at five loops [33]. An improved analysis based on harmonic superspace power-counting improved this lower limit to six loops [34]. In [11] a seven loop counterterm was proposed, but in [15] it is proposed from string perturbation theory arguments that the four graviton amplitude may be eight-loop finite. The claim in [14] is even stronger: $\mathcal{N} = 8$ supergravity may have the same degree of divergence as $\mathcal{N} = 4$ super-Yang-Mills theory and may therefore be ultraviolet finite. But no definitive calculations have been made yet to prove that claim; up to now, there is no firmly established example of a counterterm which does not arise in the effective actions but would be allowed by superspace non-renormalization theorems.

Because of all these open problems, we believe that higher order terms in $\mathcal{N} = 8$ supergravity definitely deserve further study.

5. Conclusions

In this paper, we analyzed in detail the reduction to four dimensions of the purely gravitational higher-derivative terms in the string effective actions, up to order α'^3, for heterotic and type IIA/IIB superstrings. From this analysis we have shown that in the four dimensional heterotic and type IIA string effective actions there must exist, besides the usual square of the Bel-Robinson tensor $\mathcal{W}_+^2\mathcal{W}_-^2$, a new $\mathcal{R}^4$ term given in terms of the Weyl tensor by $\mathcal{W}_+^4 + \mathcal{W}_-^4$. This new term results from the dimensional reduction of the order α'^3 effective actions, at one string loop, of these theories. By requiring four dimensional supersymmetry, this term must be, like any other, part of some superinvariant, but it had been shown, under some assumptions (conservation of chirality), that such a superinvariant could not exist by itself in pure $\mathcal{N} = 1$ supergravity. But, by taking a specific (chirality-breaking) coupling of this term to a chiral multiplet in $\mathcal{N} = 1$ supergravity, we were indeed able to obtain the desired superinvariant. The $\mathcal{W}_+^4 + \mathcal{W}_-^4$ term appeared after elimination of its auxiliary fields, by itself, without any couplings to the chiral multiplet fields.

To summarize, we have demonstrated the existence of a new $\mathcal{R}^4$ superinvariant in $d = 4$ supergravity, a result that many people would find unexpected. The supersymmetrization of this new $\mathcal{R}^4$ term in extended supergravity remains an open problem, but we found it in $\mathcal{N} = 1$ supergravity. As we concluded from our analysis of the dimensional reduction of order α'^3 gravitational effective actions, this new $\mathcal{R}^4$ term has its origin in the dimensional reduction of the corresponding term in M-theory, a theory of which there is still a lot to be understood. We believe therefore that the complete study of this term and its supersymmetrization deserves further attention in the future.

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